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The frequency response of large-amplitude oscillations of a trapped Bose condensed gas

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Abstract

We study the frequency response of large-amplitude oscillations of a trapped Bose condensed gas. On the basis of the Thomas–Fermi approximation, we deduce the hydrodynamical equation including the excitation source of the velocity drive, and obtain the analytical expression for the frequency as a function of velocity fluctuation amplitude, trap geometry, and symmetry of modes. By solving the expansion equation after switching off the trap, we find a simple relationship between the velocity amplitude and the oscillation amplitude. The theoretical calculations on the response frequencies agree well with the existing experimental observations.

1. Introduction

The realization of a Bose–Einstein condensate (BEC) in trapped atoms [1, 2] has stimulated extensive interest in the theoretical study of non-homogeneous interacting Bose gases. The lower excitation modes, in particular those for the large-amplitude response and the experimentally required ballistic expansion, have been measured [3, 4]. The theoretical predictions for the excitation spectrum are in remarkable agreement with the experiments, and confirm the quantitative validity of the non-linear Schrödinger equation for describing the evolution of the condensate wavefunction. The equation for the lower excitation modes of the trapped Bose gases has been studied in detail using analytic methods [5–7], variational methods [8, 9], and numerical methods [10–13]. Since the experiments [3, 4] were performed with an axially symmetrical harmonic trap and validated for the regime suited to hydrodynamical treatment [5], we focus purely on the system with an axial symmetry trap in the hydrodynamical limit. Fliesser *et al* [6] and Öhberg *et al* [7] obtained analytical solutions for the homogeneous hydrodynamical equation with different treatments. Ma and Chui [14] also studied the eigenvalue problem of the equation in more detail. By using the techniques of decoupling and reducing the dimension of the eigenvalue problem, we solved a coupled two-dimensional hydrodynamical equation and obtained the analytical expressions for all modes in cylindrical coordinates.

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Furthermore, Pitaevskii and Stringari [15] and Braaten and Pearson [16] calculated the correction of the excitation spectrum due to the chemical potential change associated with the density fluctuation in the hydrodynamical limit. Ma *et al* [17] calculated the correction of the excitation spectrum due to the kinetic energy enhancement and chemical potential reduction beyond the hydrodynamical limit. Nevertheless, in order to measure the excitation spectrum in experiments, first of all, one needs to excite the system to the lower excited states, meanwhile detecting the amplitude-dependent frequency shifts (including the ballistic expansion) as in [3, 4]. Theoretically, Dalfovo *et al* [18] studied the non-linear dynamics of a BEC using a new non-linear transformation and obtained the frequency response qualitatively, although the frequency shift is less by about a half than the experimental datum for the large-amplitude oscillation for the 002 mode (for the definition of the $n_z n_s m$ modes, see below). Storey and Olshanii [19] carried out an exact treatment of the problem by the trajectory method and calculated the excitation amplitude-dependent frequency shift. They obtained a good agreement with the experimental data of [3] for the 002 mode, but did not include the 010 mode from existing experimental data. Liu *et al* [20] studied the non-linear effects in the interference of the BEC using an exact solution of the one-dimensional non-linear Schrödinger equation. Li *et al* [21] studied the dynamics of a two-component BEC with a coupling drive by means of a pair of bosonic operators, and obtained the effects of the coupling drive on the evolution of the relative phase of the two-component BEC. Very recently, Ma and Chui [22] derived the elementary excitation spectrum of the BEC for any trapped atom number with the aid of a corrected sum rule and a generalized virial identity.

On the other hand, we know that it is often possible to measure the frequency shift directly to a very high degree of accuracy. Theoretical studies of the frequency response of the excitation amplitude need to extend to any excitation mode, or at least to all the existing experimental modes. Using different but simple treatments, this paper is concerned with the frequency response of large-amplitude oscillations (including the ballistic expansion) in an axially trapped interacting Bose gas. The essential features of our treatment in comparison with the previous theoretical works are as follows. Using the Thomas–Fermi approximation and considering the excitation source of the velocity drive, we first deduce the inhomogeneous hydrodynamical equation for the complete modes and then obtain the analytical expressions for the frequency responses for the 002, 010, and 200 modes. By solving the expansion equation after opening the trap, we finally obtain an explicit and simple relationship between the velocity and the oscillation amplitude response. Physically, we can get information on the frequency response of a BEC under large-amplitude oscillations and understand the concept of elementary excitations of a trapped Bose condensed gas.

2. Theoretical descriptions

The dynamics of the BEC is described by the non-linear Schrödinger equation for the macroscopic wavefunction $\Phi(\boldsymbol{r}, t)$:

$$\mathrm{i}\hbar\dot{\Phi}(\boldsymbol{r}, t) = \left[\frac{-\hbar^2}{2M}\nabla^2 + U_{ext}(\boldsymbol{r}) + g|\Phi(\boldsymbol{r}, t)|^2 - \mu\right]\Phi(\boldsymbol{r}, t) \tag{1}$$

where $U_{ext}(\boldsymbol{r}) = \frac{1}{2}M\omega_\perp^2(s^2 + \lambda^2 z^2)$ represents the axially symmetric magnetic trapping potential with harmonic oscillator frequencies $\omega_\perp$ in the xy-plane and $\lambda\omega_\perp$ in the z-axis direction, $g = 4\pi\hbar^2 a_{sc}/M$ characterizes the atom–atom interaction and is defined in terms of the s-wave scattering length a_{sc}, and the chemical potential μ is determined by the normalization of $\Phi(\boldsymbol{r}, t)$ with $N = \int |\Phi(\boldsymbol{r}, t)|^2 \,\mathrm{d}^3\boldsymbol{r}$ the atom number. In the present paper we only consider the case of the large-N limit. In this case we can neglect the kinetic

energy pressure with respective to the strong atom–atom interaction. We introduce the atom density $\rho(\boldsymbol{r},t)=|\Phi(\boldsymbol{r},t)|^2$ and velocity field $\boldsymbol{v}(\boldsymbol{r},t)=-\frac{\hbar}{M}\boldsymbol{\nabla}\phi(\boldsymbol{r},t)$ to substitute for the wavefunction $\Phi(\boldsymbol{r},t)=\sqrt{\rho(\boldsymbol{r},t)}\exp[\mathrm{i}\phi(\boldsymbol{r},t)]$. In the ground state, from $\rho(\boldsymbol{r},t)=\rho_0(\boldsymbol{r})$, $\boldsymbol{v}(\boldsymbol{r},t)=0$, and $\mu=\mu_0$, we have the Thomas–Fermi approximation solution

$$\rho_0(\boldsymbol{r})=[\mu_0-\tfrac{1}{2}M\omega_\perp^2(s^2+\lambda^2z^2)]/g. \tag{2}$$

The boundary conditions are $\rho_0(0)=\mu_0/g$ at $r=0$ and $\rho_0(R)=0$ at $r=R$. From $\mu_0=\frac{1}{2}M\omega_\perp^2R^2$, $N=\int\rho_0(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}=\frac{8\pi}{15\lambda}R^3\rho_0(0)$ and $a_{ho}=\sqrt{\hbar/M\omega_\perp}$, a harmonic oscillator characteristic length, one finds that the BEC boundary is $R=a_{ho}(\lambda P)^{1/5}$ and the ground state chemical potential is $\mu_0=\frac{1}{2}\hbar\omega_\perp(\lambda P)^{2/5}$ with $P=15Na_{sc}/a_{ho}$ proportional to the product of the interaction strength and the atom number. When the parameter P is sufficiently large, it corresponds to the Thomas–Fermi approximation. For the lower excited states, by writing $\rho(\boldsymbol{r},t)=\rho_0(\boldsymbol{r})+\delta\rho(\boldsymbol{r},t)$ and assuming the chemical potential to be constant since most of the atoms are in the ground state, one obtains equation (1) in the usual hydrodynamical form

$$M\dot{\boldsymbol{v}}(\boldsymbol{r},t)+\boldsymbol{\nabla}[g\,\delta\rho(\boldsymbol{r},t)+\tfrac{1}{2}M\boldsymbol{v}^2(\boldsymbol{r},t)]=0, \tag{3}$$

$$\dot{\rho}(\boldsymbol{r},t)+\boldsymbol{\nabla}\cdot[\rho(\boldsymbol{r},t)\boldsymbol{v}(\boldsymbol{r},t)]=0. \tag{4}$$

In the linearization of the density fluctuations $\delta\rho(\boldsymbol{r},t)$, the velocity fluctuation $\boldsymbol{v}(\boldsymbol{r},t)$ must be considered, since we are studying the frequency responses of the velocity drive. By combining equations (3) and (4) and taking into account the terms $\boldsymbol{v}^2(\boldsymbol{r},t)$ and $|\boldsymbol{v}\,\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})|$, we obtain

$$M\,\delta\ddot{\rho}+\boldsymbol{\nabla}\cdot(g\rho_0\,\boldsymbol{\nabla}\,\delta\rho)=-M\,\boldsymbol{\nabla}\cdot[\boldsymbol{v}\,\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{v})+\tfrac{1}{2}\rho_0\boldsymbol{v}^2]. \tag{5}$$

Using the ansatz $\delta\rho(\boldsymbol{r},t)=\delta\rho(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega t}$ and $\boldsymbol{v}(\boldsymbol{r},t)=\boldsymbol{v}(\boldsymbol{r})\mathrm{e}^{\mathrm{i}\omega t/2}$, equation (5) simplifies to

$$\gamma\,\delta\bar{\rho}(\boldsymbol{r})+\tfrac{1}{2}\,\bar{\boldsymbol{\nabla}}\cdot[(1-\bar{s}^2-\lambda^2\bar{z}^2)\,\bar{\boldsymbol{\nabla}}\,\delta\bar{\rho}(\boldsymbol{r})]=F(\boldsymbol{r}), \tag{6}$$

$$F(\boldsymbol{r})\equiv\bar{\boldsymbol{\nabla}}\cdot\{\bar{\boldsymbol{v}}(\boldsymbol{r})\,\bar{\boldsymbol{\nabla}}\cdot[(1-\bar{s}^2-\lambda^2\bar{z}^2)\bar{\boldsymbol{v}}(\boldsymbol{r})]+\tfrac{1}{2}(1-\bar{s}^2-\lambda^2\bar{z}^2)\,\bar{\boldsymbol{\nabla}}\bar{\boldsymbol{v}}^2(\boldsymbol{r})\} \tag{7}$$

where $\bar{\boldsymbol{\nabla}}\equiv R\,\boldsymbol{\nabla}$ is the dimensionless gradient operator, $\delta\bar{\rho}(\boldsymbol{r})\equiv\delta\rho(\boldsymbol{r})/\rho_0(0)$ is the dimensionless density fluctuation, $\bar{\boldsymbol{v}}(\boldsymbol{r})\equiv\boldsymbol{v}(\boldsymbol{r})\sqrt{M/g\rho_0(0)}$ is the dimensionless velocity fluctuation, and $\gamma\equiv\omega^2/\omega_\perp^2$ is the squared frequency relative to the trapping frequency. The dimensionless linear equation (6) (including the velocity fluctuation) is the hydrodynamical equation for any excitation mode. The inhomogeneous $F(\boldsymbol{r})$ term represents the velocity drive. For a certain excited mode, the velocity fluctuation will take a fixed form. Without the velocity fluctuation, equation (6) reduces to the usual eigenvalue equation for determining the elementary excitation spectrum.

Before solving equation (6), we first briefly review its eigensolutions [14] with $\boldsymbol{v}(\boldsymbol{r})\equiv0$ or $F(\boldsymbol{r})\equiv0$. In this special case, the density fluctuation takes the form $\delta\rho^{(0)}_{n_zn_sm}(\boldsymbol{r})=\bar{s}^{|m|}P^{(2n_s)}_{n_p}(\bar{s},\bar{z})\mathrm{e}^{\mathrm{i}m\varphi}$ with a coupled polynomial

$$P^{(2n_s)}_{n_p}(\bar{z},\bar{s})=\sum_{k=0}^{n_p}\sum_{n=0}^{\mathrm{int}[k/2]}b_{k,n}\bar{z}^{k-2n}\bar{s}^{2n}. \tag{8}$$

We had found a principal quantum number n_p for the 'total energy', $n_p=0,1,2,\ldots$. m is the azimuthal quantum number for the axial component of angular momentum $m=0,\pm1,\pm2,\ldots$. n_s is the radial quantum number for the series of numbers of modes $2n_s$ at the fixed numbers n_p and m, $n_s=0,1,2,\ldots,\mathrm{int}[n_p/2]$. n_z is the axial quantum number for the corresponding mode number, $n_z=n_p-2n_s$, i.e., $n_z=0,2,4,\ldots,n_p$ for even n_p and $n_z=1,3,5,\ldots,n_p$ for odd n_p. The entire set of elementary excitation modes are labelled by three quantum numbers, n_z, n_s, and m (we have denoted them as n_zn_sm modes). The coefficient relation of

$b_{k,n}$ is

$$4(n+1)(n+|m|+1)b_{k+2,n+1}+(k-2n+2)(k-2n+1)b_{k+2,n}$$
$$=4\lambda^2(n+1)(n+|m|+1)b_{k,n+1}-[2\gamma-2|m|-4n(n+|m|+1)$$
$$-\lambda^2(k-2n)(k-2n+1)]b_{k,n}+(k-2n+2)(k-2n+1)b_{k,n-1}. \tag{9}$$

Using the decoupling and dimension reduction techniques and taking $k=n_p$ in equation (9), the corresponding eigenvalue is the solution of

$$-1=\cfrac{g_0}{1+\cfrac{g_1}{1+\cfrac{g_2}{1+\cdots\cdots\cdots}}}, \tag{10}$$
$$\ddots$$
$$1+\frac{g_{Z-2}}{1+0}$$

with $Z=1+\mathrm{int}[n_p/2]$ and

$$g_n=\frac{-4\lambda^2(n+1)(n+|m|+1)(n_p-2n-1)(n_p-2n)}{2\gamma-2|m|-4n(n+|m|+1)-\lambda^2(n_p-2n)(n_p-2n+1)}$$
$$\times[2\gamma-2|m|-4(n+1)(n+|m|+2)-\lambda^2(n_p-2n-2)(n_p-2n-1)]^{-1}. \tag{11}$$

Equation (10) has Z different solutions for $\omega^2_{n_zn_sm}$ which have been labelled by the quantum numbers n_s for $n_s=0,1,2,\ldots,\mathrm{int}[n_p/2]$.

The existence of an axial parity $\boldsymbol{P}_z$, i.e., $\boldsymbol{P}_z=[+]$ ($[-]$) for even (odd) n_z, leads to the coefficients $b_{k,n}$ in equation (9) being proportional only to a density fluctuation amplitude $b_{0,0}$ ($b_{1,0}$). We know that $b_{0,0}$ ($b_{1,0}$) $\to 0$ in the homogeneous linear elementary excitation treatment. However, in order to measure the excitation spectrum, one must apply a finite fluctuation amplitude experimentally and theoretically. One can also investigate the condensate response for different driving amplitudes. In this work, the velocity drive leads to the density fluctuation and is characterized by the inhomogeneous term of equation (6). We assume that the symmetry of the n_zn_sm modes remains unchanged in all response processes.

Now we can solve equation (6) exactly for lower excitation modes. For the 002 mode, we know the oscillator frequency $\omega_{002}(v=0)=\sqrt{2}\omega_\perp$ and density fluctuation $\delta\rho^{(0)}_{002}=b_{0,0}\bar{s}^2\exp(\mathrm{i}\,2\varphi)$ in the absence of the velocity drive ($v=0$). In order to excite this radial oscillation, we choose the driving velocity $\bar{v}_{002}(\boldsymbol{r})=u\bar{s}\exp(\mathrm{i}\varphi)$ with u the relative velocity fluctuation magnitude. The expression for $F(\boldsymbol{r})$ in equation (7) becomes $F_{002}(\boldsymbol{r})=-10u^2\bar{s}^2\exp(\mathrm{i}\,2\varphi)$, and hence equation (6) has a solution of the form $\delta\bar{\rho}_{002}(\boldsymbol{r})=b\bar{s}^2\exp(\mathrm{i}\,2\varphi)$ with b the relative density fluctuation magnitude. The corresponding response frequency is

$$\omega_{002}(u)=\sqrt{2}(1+5u^2/b)^{1/2}\omega_\perp. \tag{12}$$

Which describes an oscillator with a forced term proportional to u^2/b.

For the 010 and 200 modes, from equation (10) we know that $\omega^2_{n_zn_s0}(v=0)=[2+\frac{3}{2}\lambda^2\pm\sqrt{4-4\lambda^2+\frac{9}{4}\lambda^4}]\omega^2_\perp$, where + (−) represents the 200 (010) mode for $\lambda\leqslant 1$, while for $\lambda\geqslant 1$, + (−) represents the 010 (200) mode. From equation (9) we also know ($n_z=0,2,n_s=1,0$) that $\delta\rho^{(0)}_{n_zn_s0}(\boldsymbol{r})=b_{0,0}(1+a_1\bar{s}^2+a_2\bar{z}^2)$ with $a_1=\gamma(0)/[2-\gamma(0)]$, $a_2=\gamma(0)[\gamma(0)-4]/[2-\gamma(0)]$, and $\gamma(u)\equiv\omega^2_{n_zn_s0}/\omega^2_\perp$. In those excitations, the system oscillations are along the z-axis, and the axial and radial oscillations have relative amplitude $a_2/a_1=\gamma(0)-4\equiv a$. Hence the driving velocity can be chosen as $\bar{v}_{n_zn_s0}(\boldsymbol{r})=u(\bar{s}+a\bar{z})$ and the expression for $F(\boldsymbol{r})$ in equation (7) becomes

$$F_{n_z n_s 0}(\boldsymbol{r}) = u^2[(3+2a+2a^2) - (13+6a+2a^2)\bar{s}^2 - 3(1+2a+4a^2)\lambda^2\bar{z}^2]. \tag{13}$$

Furthermore, equation (6) for those modes has a solution of the form $\delta\bar{\rho}_{n_z n_s 0}(\boldsymbol{r}) = b(1+b_1\bar{s}^2 + b_2\bar{z}^2)$ with $b_1 = [\gamma - 2(5+2a)u^2/b]/(2-\gamma)$ and $b_2 = [\gamma - 2a(2+5a)u^2/b]/(2\lambda^2-\gamma)$. The associated frequency $\gamma = \gamma(u)$ satisfies the cubic equation

$$\begin{aligned}\gamma^2 - (4+3\lambda^2)\gamma + 10\lambda^2 = (u^2/b)\{&4(7+4a+3a^2)\lambda^2/\gamma \\ &+ 2[3\lambda^2 - 7 - 2a + (2-3\lambda^2)a^2] - (3+2a+2a^2)\gamma\}.\end{aligned} \tag{14}$$

The frequency responses depending on u^2/b are weak due to the coupling between the monopole and quadrupole oscillations in the 010 and 200 modes.

3. Results and discussion

In order to make a comparison between the theoretical calculations of equations (12) and (14) and the experimental data in [3, 4], it is necessary to simulate the atomic cloud expansion, since the response frequency is measured by imaging the atomic cloud after opening the trap and allowing an expansion time of a few milliseconds. The trap is opened when starting the expansion and the velocity fluctuation vanishes during the expansion. From $U_{ext}(\boldsymbol{r}) = 0$ and $v = 0$ in equation (5), we get the expansion equation

$$M\,\delta\ddot{\rho}_e + g\rho_0(0)\,\nabla^2\,\delta\rho_e = 0 \tag{15}$$

due to the atom–atom interaction. Taking the ansatz $\delta\rho_e(\boldsymbol{r},t) = \delta\rho_e(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\sqrt{\gamma}\omega_\perp t}$ with the harmonic oscillator frequency $\sqrt{\gamma}\omega_\perp = \omega = \omega_{n_z n_s m}$, the wave motion equation (15) simplifies to

$$\bar{\nabla}^2\,\delta\rho_e(\boldsymbol{r}) + 2\gamma\,\delta\rho_e(\boldsymbol{r}) = 0. \tag{16}$$

This is a Helmholtz equation, valid for any excitation mode. In the axial symmetry case, we can separate the variables and then take the form $\delta\rho_e(\boldsymbol{r}) = \rho_0(0)b\bar{s}^m\mathrm{e}^{-k\bar{z}+\mathrm{i}m\varphi}f_e(\bar{s})$. When k is a constant and $\mathrm{Re}\,k \geqslant 0$ in a physical situation, the relative density fluctuation amplitude b maintains its original value from before the expansion. The radial function $f_e(\bar{s})$ satisfies the equation

$$f_e'' + \frac{1+2m}{\bar{s}}f_e' + (2\gamma + k^2)f_e = 0. \tag{17}$$

The solution of equation (17) is a first-kind Bessel function of order m, namely $f_e(\bar{s}) = J_m(\sqrt{2\gamma+k^2}\bar{s})$.

The values of k are determined as the limit values of $\langle\bar{s}^2\rangle$ with respect to k. Here $\langle\bar{s}^2\rangle$ is the mean squared width of the condensate in the radial coordinate. The experimental oscillation amplitude σ of the condensate corresponds theoretically to

$$\sigma = \frac{\langle\bar{s}^2\rangle_{max}^{1/2} - \langle\bar{s}^2\rangle_{min}^{1/2}}{\langle\bar{s}^2\rangle_{max}^{1/2} + \langle\bar{s}^2\rangle_{min}^{1/2}}. \tag{18}$$

We next calculate $\langle\bar{s}^2\rangle$. From $\rho(\boldsymbol{r},t) = \rho_0(\boldsymbol{r}) + \delta\rho_e(\boldsymbol{r},\boldsymbol{t})$ with $\delta\rho_e(\boldsymbol{r}) \ll \rho_0(\boldsymbol{r})$, $N = \int\rho_0(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}$, and $\int\delta\rho_e(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r} = 0$, we know that the volume integral is confined within the spheroid given by $\lambda|\bar{z}| \leqslant \sqrt{1-\bar{s}^2}$. The calculation of $\langle\bar{s}^2\rangle$ from the expression $\langle\bar{s}^2\rangle = 4\pi\frac{R^3}{N}\int_0^1\bar{s}^3\,\mathrm{d}\bar{s}\int_0^{\sqrt{1-\bar{s}^2}/\lambda}\mathrm{d}\bar{z}\,[\rho_0(\boldsymbol{r}) + \delta\rho_e(\boldsymbol{r},\boldsymbol{t})]$ gives

$$\langle\bar{s}^2\rangle = \tfrac{2}{7} + \tfrac{15}{4}\bar{u}^2 b\cos(\sqrt{\gamma}\omega_\perp t)\,f_m(k) \tag{19}$$

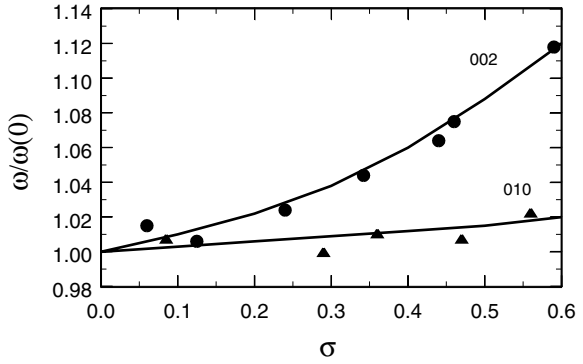

Figure 1. The normalized excitation spectrum $\omega(\sigma)/\omega(0)$ as a function of the oscillation amplitude σ for the 002 and 010 modes in the axially symmetric harmonic potential traps. The dots and triangles represent the corresponding experimental data for $\lambda = \sqrt{8}$ [3].

with

$$f_m(k) \equiv \frac{2\lambda}{k}\int_0^1 \mathrm{d}\bar{s}\,(1-\mathrm{e}^{-(k/\lambda)\sqrt{1-\bar{s}^2}})\bar{s}^{m+3}J_m\left(\sqrt{2\gamma+k^2}\bar{s}\right)$$
$$= \sum_{i=1}^{\infty}\frac{\Gamma(i/2+1)}{i!}\left(-\frac{k}{\lambda}\right)^{i-1}\sum_{j=0}^{\infty}\frac{(-1)^j(m+j+1)}{j!\Gamma(m+i/2+j+3)}\left(\frac{2\gamma+k^2}{4}\right)^{j+m/2}$$
$$= \sum_{i=0}^{\infty}\left(\frac{k}{\lambda}\right)^{2i}\sum_{j=0}^{\infty}\frac{(-1)^j(m+j+1)}{j!}\left(\frac{2\gamma+k^2}{4}\right)^{j+m/2}$$
$$\times\left[\frac{2^{m+j+2}}{(2i)!(2m+2i+2j+5)!!}-\frac{(i+1)!k}{(2i+2)!(m+i+j+3)!\lambda}\right]. \tag{20}$$

In our deduction we have used the series expression for $J_m(x) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!(m+j)!}(\frac{x}{2})^{2j+m}$ and the integral $\int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x = \Gamma(p)\Gamma(q)/\Gamma(p+q)$.

By substituting the experimental ratio of the axial to radial trap frequencies $\lambda = \sqrt{8}$ and the normalized frequencies $\sqrt{\gamma} \approx \sqrt{\gamma(0)} = \sqrt{2}$ (3.23) for the 002 (010) mode in equation (12) (equation (14)) together with $u = 0$ into (20), we have numerically the limit values

$$f_2^{\max}(0.0) = 0.138, \qquad f_2^{\min}(4.3) = -0.048, \tag{21}$$
$$f_0^{\max}(0.0) = 0.225, \qquad f_0^{\min}(6.4) = -0.083. \tag{22}$$

Figure 1 shows the experimental data (dots and triangles) from [3] for the normalized excitation spectrum $\omega(\sigma)/\omega(0)$ as a function of the oscillation amplitude σ for the 002 and 010 modes (respectively). These points were obtained with radial trap frequencies $\omega_\perp/2\pi$ of 132 Hz and 129 Hz and an expansion time of 7 ms [3]. The number of condensed atoms was approximately 4500 and is sufficiently large that the Thomas–Fermi approximation is valid. In the absence of experimental values of the relative density fluctuation amplitude b, we take $b = 5.3$ and $b = -3.2$ for the 002 and 010 modes. We would like to note that for the 010 mode the relative amplitude $a = -0.77 < 0$ in $\bar{\boldsymbol{v}}_{010}(\boldsymbol{r}) = u(\bar{s} + a\bar{z})$ and the coefficient $b_1 < 0$ in $\delta\bar{\rho}_{010}(\boldsymbol{r}) = b(1 + b_1\bar{s}^2 + b_2\bar{z}^2)$, so we have to choose $b < 0$. The theoretical curves, calculated for the experimental parameters and shown in figure 1, agree well with the experimental observations. Since $u^2 \ll 1$, the normalized mode excitation spectrum in equations (11)

and (14) simplifies to

$$\frac{\omega(\sigma)}{\omega(0)} = \sqrt{1 + \frac{\sigma/b^2}{c_1 - c_2\sigma}}, \tag{23}$$

with $c_1 = 0.121$ (1.17) and $c_2 = 0.059$ (-0.541) for the 002 (010) mode.

Finally, let us make concluding remarks.

(i) The theoretical elementary excitation corresponds to $\delta\rho^{(0)}(\boldsymbol{r}) \propto b_{0,0}$ ($b_{1,0}$) $\to 0$, while the experimental excitation corresponds to $\delta\rho(\boldsymbol{r}) \ll \rho_0(\boldsymbol{r})$, and hence the relative fluctuation amplitudes, u and b, are finite.

(ii) From $N = \int[\rho_0(\boldsymbol{r}) + \delta\rho(\boldsymbol{r}, t)]\,\mathrm{d}^3\boldsymbol{r}$ we can determine b as a function of N. For simplicity, we have chosen $\int \delta\rho(\boldsymbol{r}, t)\,\mathrm{d}^3\boldsymbol{r} = 0$, so b becomes a control parameter. However, we are able to determine its sign for modes of a certain kind.

(iii) The BEC occurs near the centre of magnetic traps and the volume integral mainly contributes in the middle region of the traps. So the linearization condition $\delta\bar{\rho}_{002}(\boldsymbol{r}) \ll \rho_0(\boldsymbol{r})$ leads to $b \ll \bar{s}^{-2} \sim 10^2$.

(iv) We find in figure 1 that the frequency of the 002 mode increases with increasing oscillation amplitude, while the frequency of the 010 mode shows a little increase. This behaviour is obvious, as shown in equation (23), both for the values of c_1 and for the sign of c_2 due to the coupling between modes.

(v) The key result of equation (23) is easy to generalize to all excitations that could be studied in future experiments.

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